Quantum Hurwitz numbers and Macdonald polynomials*

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Abstract

Parametric families in the center $\mathbf{Z}(\mathbf{C}[S_n])$ of the group algebra of the symmetric group are obtained by identifying the indeterminates in the generating function for Macdonald polynomials as commuting Jucys-Murphy elements. Their eigenvalues provide coefficients in the double Schur function expansion of 2D Toda τ-functions of hypergeometric type. Expressing these in the basis of products of power sum symmetric functions, the coefficients may be interpreted geometrically as parametric families of quantum Hurwitz numbers, enumerating weighted branched coverings of the Riemann sphere. Combinatorially, they give quantum weighted sums over paths in the Cayley graph of S_n generated by transpositions. Dual pairs of bases for the algebra of symmetric functions with respect to the scalar product in which the Macdonald polynomials are orthogonal provide both the geometrical and combinatorial significance of these quantum weighted enumerative invariants.

Contents

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1 Introduction: weighted Hurwitz numbers

A new method for constructing parametric families of 2D Toda τ-functions [28–30] of hypergeometric type [26] that serve as generating functions for various types of weighted Hurwitz numbers was developed in [10–13, 15]. This was originally inspired by the work of Pandharipande [27] and Okounkov [25], which first used a special case of KP and 2D-Toda τ-functions as generating functions for single and double for Hurwitz numbers when all branchings other than the ones specified at one or two points rare equired to be simple, and the weighting for these is uniform. The general case gives infinite parametric families of weighted enumerations of n-fold branched coverings of the Riemann sphere or, equivalently, weighted paths in the Cayley graph of the symmetric group S_n generated by transpositions. They are derived from parametric families of weight generating functions by defining associated symmetric functions of an arbitrary number of indeterminates multiplicatively. Replacing one set of indeterminates in the Cauchy-Littlewood generating function [21] by the commuting elements of the group algebra introduced by Jucys [16] and Murphy [22], while evaluating the other set at parameter values defining the weightings provides parametric families of elements of the center $\mathbf{Z}(\mathbf{C}[S_n])$ of the group algebra. Expanding these as sums over products of dual bases of the algebra of symmetric functions, and applying them multipicatively to the basis of the center of the group algebra consisting of cycle-type sums $\{C_\mu\}$ leads to an identification of both the geometrical significance of the weighted Hurwitz numbers and the combinatorial one.

It was shown in [10, 11, 15] that all previously studied examples of generating functions for Hurwitz numbers [1–4, 8, 9, 17, 25, 27, 31] may be viewed as special cases of this general construction, and several new examples were introduced, including three forms of quantum Hurwitz number [11] and their multispecies generalization [12]. Other notions of weighted or quantum Hurwitz numbers have also been considered, including those for branched coverings of $\mathbf{RP}^2$, whose generating functions are BKP τ-functions [23, 24], and Hurwitz numbers enumerating factorization of Singer cycles [20].

In the following, we extend the special class of weighted Hurwitz numbers introduced in [10, 11] by introducing an additional pair (q, t) of deformation parameters in the definition

of the weight generating functions. The result is to replace the Cauchy-Littlewood formula, which generates dual bases in the algebra of the symmetric functions with respect to the standard scalar product pairing by the corresponding one for MacDonald polynomials [21]. In Section 2 the general method is developed, and used to derive an infinite parametric family of 2D Toda τ-functions of hypergeometric type depending not only on the previously introduced classical weight determining parameters, but also the additional pair (q, t) of quantum deformation parameters entering in the definition of the scalar product. These are shown to be generating functions for infinite parametric families of quantum weighted Hurwitz numbers when expanded in the basis of products of power sum symmetric functions. The combinatorial significance, in terms of quantum weighted paths in the Cayley graph, is derived in Theorem 2.2, and the geometric one, in terms of quantum weighted enumeration of branched covers, in Theorem 2.5. Section 3 is devoted to various examples obtained by specialization of the parameters and taking limits. It is shown how all previously studied cases of weighted Hurwitz numbers, whether classical or quantum, may be recovered within this more general setting, and a number of new families are added, including those associated to Hall-Littlewood and Jack polynomials.

2 2D Toda τ-functions, MacDonald polynomials and quantum weighted Hurwitz numbers

2.1 The generating function for Macdonald polynomials

Following [21] (sec. VI. 2), we define a 2-parameter family of scalar products $(\ ,\ )_{(q,t)}$ on the algebra Λ of symmetric functions in an infinite number of indeterminates $\mathbf{x} = (x_1, x_2, \dots)$, such that the power sum symmetric functions are orthogonal;

$$(p_\lambda, p_\mu)_{(q,t)} := z_\mu(q,t)\delta_{\mu,\nu} \tag{2.1}$$

where

$$p_\lambda := \prod_{i=1}^{\ell(\lambda} p_{\lambda_i} \in \Lambda, \qquad p_j := \sum_i x_i^j, \quad j \in \mathbf{N} \tag{2.2}$$

are the power sum symmetric functions corresponding to the integer partition

$$\lambda = \lambda_1 \geq \cdots \geq \lambda_{\ell(\lambda)} \tag{2.3}$$

of length $\ell(\lambda)$. The normalization factor $z_\mu(q,t)$ is defined as

$$z_\mu(q,t) := z_\mu n_\mu(q,t), \quad z_\mu := \prod_{i=1}^{\mu} i^{m_i(\mu)}(m_i(\mu))!. \tag{2.4}$$

where $m_i(\mu)$ is the number of parts of μ equal to i and

$$n_\mu(q,t) := \prod_{i=1}^{\ell(\mu)} \frac{1-q^{\mu_i}}{1-t^{\mu_i}}. \tag{2.5}$$

The Macdonald polynomials $\{P_\lambda(\mathbf{x},q,t)\}$ may be defined [21, Chapt. VI] as the unique basis for Λ determined by two conditions: orthogonality with respect to the scalar product $(\ ,\ )_{(q,t)}$

$$(P_\lambda, P_\mu)_{(q,t)} = 0 \quad \text{if } \lambda \neq \mu, \tag{2.6}$$

and lower triangular normalized decomposition (with respect to the dominance partial ordering [21, Sec. I.1, pg. 7]) in the basis $\{m_\lambda\}$ of monomial symmetric functions

$$P_\lambda(\mathbf{x},q,t) = m_\lambda + \sum_{\mu<\lambda} K_{\lambda\mu}(q,t) m_\mu(\mathbf{x}). \tag{2.7}$$

The generating function [21]

$$\Pi(\mathbf{x},\mathbf{y},q,t) := \prod_{ij} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty} \tag{2.8}$$

where

$$(u;q)_\infty := \prod_{k=0}^{\infty}(1-uq^k) \tag{2.9}$$

is the (infinite) quantum Pochhammer symbol, has the following alternative expansions [21, Sec. VI.2] in terms of products of symmetric functions in the pair of infinite sequences of determinate $\mathbf{x}=(x_1,x_2,\dots)$, $\mathbf{y}=(y_1,y_2,\dots)$,

$$\Pi(\mathbf{x},\mathbf{y},q,t) = \sum_\lambda b_\lambda(q,t) P_\lambda(\mathbf{x},q,t) P_\lambda(\mathbf{y},q,t) \tag{2.10}$$

$$= \sum_\lambda g_\lambda(\mathbf{x},q,t)\, m_\lambda(\mathbf{y}) \tag{2.11}$$

$$= \sum_\lambda m_\lambda(\mathbf{x})\, g_\lambda(\mathbf{y},q,t), \tag{2.12}$$

where

$$b_\lambda(q,t) := (P_\lambda, P_\lambda)^{-1}_{(q,t)} \tag{2.13}$$

and

$$g_\lambda(\mathbf{x},q,t) := \prod_{i=1}^{\ell(\lambda)} g_{\lambda_i}(\mathbf{x},q,t), \tag{2.14}$$

where

$$g_j(\mathbf{x},q,t) := b_{(j)}(q,t) P_{(j)}(\mathbf{x},q,t) = \sum_{\mu,\,|\mu|=j} z_\mu(q,t)^{-1} p_\mu(\mathbf{x}). \tag{2.15}$$

The basis $\{g_\lambda(\mathbf{x}, q, t)$ provides the (q,t) analog of the elementary $\{e_\lambda\}$ and complete $\{h_\lambda\}$ symmetric function basis [21], interpolating between them in the case of Hall polynomials $(q = 0)$.

2.2 Quantum weight generating function

We now proceed as in [10, 11] to define parametric families within the center $\mathbf{Z}(\mathbf{C}[S_n])$ of the group algebra $\mathbf{C}[S_n]$ by identifying the indeterminates $(x_1, x_2, \dots)$ with a given set of constants $(c_1, c_2, \dots)$ and the $(y_1, y_2, \dots)$ with z times the commuting Jucys-Murphy elements $\mathcal{J} := (\mathcal{J}_1, \dots, \mathcal{J}_n)$ of $\mathbf{C}[S_n]$ [5, 16, 22], defined as :

$$\mathcal{J}_b \coloneqq \sum_{a=1}^{b-1} (a\, b), \quad b = 1, \dots, n, \quad n \in \mathbf{N}^+. \tag{2.16}$$

We define the *quantum weight generating function* as

$$M(q, t, \mathbf{c}, z) := \prod_{i=1}^{\infty} M(q, t, zc_i) = \sum_{j=0}^{\infty} g_j(\mathbf{c}, q, t) z^j. \tag{2.17}$$

where

$$M(q, t, z) := \frac{(tz; q)_\infty}{(z; q)_\infty} = \prod_{k=0}^{\infty} \frac{1 - tzq^k}{1 - zq^k}. \tag{2.18}$$

The Jucys-Murphy elements generate a commuting subalgebra of the group algebra $\mathbf{C}[S_n]$, and any symmetric polynomial in them is in the center $\mathbf{Z}(\mathbf{C}[S_n])$, The resulting central element $M_n(q, t, \mathbf{c}, z\mathcal{J}) \in \mathbf{Z}(\mathbf{C}[S_n])$ is

$$M_n(q, t, \mathbf{c}, z\mathcal{J}) := \prod_{a=1}^{n} M(q, t, \mathbf{c}, z\mathcal{J}_a) = \Pi(\mathbf{c}, z\mathcal{J}, q, t) \tag{2.19}$$

$$= \sum_{\lambda} z^{|\lambda|} g_\lambda(\mathbf{c}, q, t) m_\lambda(\mathcal{J}) \tag{2.20}$$

$$= \sum_{\lambda} z^{|\lambda|} m_\lambda(\mathbf{c}) g_\lambda(\mathcal{J}, q, t). \tag{2.21}$$

2.3 Bases for $\mathbf{Z}(\mathbf{C}[S_n])$ and the eigenvalues of $M_n(q, t, \mathbf{c}, z\mathcal{J})$

Proceeding as in [10, 11, 15], we make use of two standard bases of $\mathbf{Z}(\mathbf{C}[S_n])$, both labelled by partitions of n. The first consists of the cycle-type sums $\{C_\mu\}$:

$$C_\mu = \sum_{h \in \mathrm{cyc}(\mu)} h, \tag{2.22}$$

where $\mathrm{cyc}(\mu) \subset S_n$ denotes the conjugacy class consisting of elements whose cycle lengths are equal to the parts μ_i of the partition μ.

The second consists of the orthogonal idempotents $\{F_\lambda\}$, corresponding to the irreducible representations of S_n, labelled by partitions λ of weight $|\lambda| = n$. These are linearly related to the cycle-type sums through the equivalent, under the characteristic map, of the Frobenius character formula [6, 7]

$$F_\lambda = h_\lambda^{-1} \sum_{\mu,\, |\mu|=|\lambda|} \chi_\lambda(\mu) C_\mu, \quad C_\mu = z_\mu^{-1} \sum_{\lambda,\, \lambda|=|\mu|} \chi_\lambda(\mu) h_\lambda F_\lambda. \tag{2.23}$$

Here $\chi_\lambda(\mu)$ is the character of the irreducible representation of Young symmetry type λ, evaluated on the class of cycle type μ, and

$$h_\lambda := \det \left(\frac{1}{(\lambda_i - i + j)!} \right)^{-1} \tag{2.24}$$

is the product of the hook lengths of the Young diagram corresponding to the partition λ.

The elements F_λ satisfy the orthogonality relations

$$F_\lambda F_\mu = F_\lambda \, \delta_{\lambda\mu}, \tag{2.25}$$

which imply that all elements of $\mathbf{Z}(\mathbf{C}[S_n])$ act diagonally under multiplication in this base. Eqs. (2.23) and (2.25) imply

$$C_\mu F_\lambda = \frac{h_\lambda \chi_\lambda(\mu)}{z_\mu} F_\lambda, \tag{2.26}$$

which means that the eigenvalue of C_μ on the basis element F_λ is the *central character*

$$\phi_\lambda(\mu) := \frac{h_\lambda \chi_\lambda(\mu)}{z_\mu}. \tag{2.27}$$

It is a basic property [5, 16, 22] that the eigenvalues of any central element $G(\mathcal{J}) \in \mathbf{Z}(\mathbf{C}[S_n])$ formed from a symmetric function $G \in \Lambda$ by identifying the indeterminates with the Jucys-Mulrphy elements are given by evaluating G on the content $\{j-i\}_{(i,j)\in\lambda}$ of the partition λ

$$G(\mathcal{J}) F_\lambda = G(\{j-i\}) F_\lambda, \quad (i,j) \in \lambda. \tag{2.28}$$

In particular, if $G(\mathbf{x})$ is formed from a product of the same expression in each of the indeterminates $\mathbf{x} = (x_1, x_2, \dots)$

$$G(\mathbf{x}) = \prod_i g(x_i), \tag{2.29}$$

the eigenvalues of $G(\mathcal{J})$ are given by the content product formula

$$G(\mathcal{J}) F_\lambda = \prod_{(i,j)\in\lambda} g(j-i) F_\lambda. \tag{2.30}$$

It follows that the eigenvalues $r_\lambda^{M(q,t,\mathbf{c},z)}$ of $M_n(q,t,\mathbf{c},z\mathcal{J})$ are given by the content product formula obtained from evaluation of the generating function $M(q,t,\mathbf{c},z)$ at $\{z(j-i)\}$

$$M_n(q,t,\mathbf{c},z\mathcal{J})F_\lambda = r_\lambda^{M(q,t,\mathbf{c},z)}F_\lambda, \tag{2.31}$$

$$r_\lambda^{M(q,t,\mathbf{c},z)} = \prod_{(i,j)\in\lambda} M(q,t,\mathbf{c},z(j-i)) = \prod_{(i,j)\in\lambda}\prod_{k=1}^{\infty}\frac{(tz(j-i)c_k;q)_\infty}{(z(j-i)c_k;q)_\infty}. \tag{2.32}$$

More generally, for an arbitrary integer $N\in\mathbf{Z}$, we define

$$r_\lambda^{M(q,t,\mathbf{c},z)}(N) := r_0^{M(q,t,\mathbf{c},z)}(N)\prod_{(i,j)\in\lambda} M(q,t,\mathbf{c},z(N+j-i)), \tag{2.33}$$

where

$$r_0^{M(q,t,\mathbf{c},z)}(N) := \prod_{j=1}^{N-1} M(q,t,\mathbf{c},(N-j)z), \quad r_0^{M(q,t,\mathbf{c},z)}(0) := 1,$$

$$r_0^{M(q,t,\mathbf{c},z)}(-N) := \prod_{j=1}^{N} M^{-1}(q,t,\mathbf{c},(j-N)z), \quad N\geq 1, \tag{2.34}$$

and hence

$$r_\lambda^{M(q,t,\mathbf{c},z)} = r_\lambda^{M(q,t,\mathbf{c},z)}(0). \tag{2.35}$$

2.4 The 2D Toda τ-function $\tau^{M(q,t,\mathbf{c},z)}(N,\mathbf{t},\mathbf{s})$ as generating function for double quantum Hurwitz numbers $F^d_{M(q,t,\mathbf{c})}(\mu,\nu)$, $H^d_{M(q,t,\mathbf{c})}(\mu,\nu)$

The general theory [14, 26, 28–30] implies that the following diagonal double Schur function expansion

$$\tau^{M(q,t,\mathbf{c},z)}(N,\mathbf{t},\mathbf{s}) := \sum_\lambda r_\lambda^{M(q,t,\mathbf{c},z)}(N)s_\lambda(\mathbf{t})s_\lambda(\mathbf{s}), \tag{2.36}$$

defines a 2D Toda τ-function of hypergeometric type, where

$$\mathbf{t} = (t_1,t_2,\dots), \quad \mathbf{s} = (s_1,s_2,\dots) \tag{2.37}$$

are the 2D Toda flow variables, which may be identified in this notation in terms of the power sums

$$t_i = \frac{p_i}{i}, \quad s_i = \frac{p'_i}{i} \tag{2.38}$$

in two independent sets of variables.

We now apply the procedure developed in [11] for deriving both the geometrical and combinatorial versions of weighted Hurwitz numbers associated to a weight generating function $G(z)$. Recall that the pure Hurwitz numbers $H(\mu^1,\dots\mu^{(k)})$ may be viewed either as the

number of n-sheeted branched coverings of the Riemann sphere having k branch points with ramification profiles given by the partitions $\{\mu^{i)}\}_{i=1,\dots k}$ of lengths $|\mu^{(i)}| = n$, weighted by the inverse of the order of the automorphism group or, equivalently, as the number of ways in which the identity element in S_n can be expressed as a product of elements belonging to the conjugacy classes $\{\mathrm{cyc}(\mu^{i)}\}$. A convenient way to express the latter is through the formula

$$H(\mu^{(1)}, \dots, \mu^{(k)}) = \frac{1}{n!}[\mathrm{Id}] \prod_{i=1}^{k} C_{\mu^{(i)}}, \tag{2.39}$$

where [Id] means taking the component of the identity element within the cycle sum basis $\{C_\mu\}$ of $\mathcal{Z}(\mathbf{C}[S_n])$ or, more generally,

$$\prod_{i=1}^{k} C_{\mu^{(i)}} = \sum_{\nu,\, |\nu|=|\mu^{(i)}|} H(\mu^{(1)}, \dots, \mu^{(k)}, \nu)\, z_\nu C_\nu, \tag{2.40}$$

which is equivalent to the Frobenius-Schur formula (see [18, Appendix A]) as shown in [11, Sec. 5.2])

$$H(\mu^{(1)}, \dots, \mu^{(k)}) = \sum_{\lambda} h_\lambda^{k-2} \prod_{i=1}^{k} \frac{\chi_\lambda(\mu^{(i)})}{z_{\mu^{(i)}}}, \tag{2.41}$$

Following [11, 15], we now consider two notions of quantum weighted Hurwitz numbers associated to the generating function (2.17): combinatorial and geometrical.

2.4.1 Combinatorial quantum weighted Hurwitz numbers [11]

Definition 2.1. *Signature of paths [11].* For every d-step path in the Cayley graph of S_n generated by transpositions, (a, b), $a < b$, starting at the conjugacy class $\mathrm{cyc}(\nu)$ and ending at the class $\mathrm{cyc}(\mu)$, define its *signature* λ as the partition of weight $|\lambda| = d$ whose parts are equal to the number of times a particular second element $b = 1, \dots, n$ appears amongst the sequence of transpositions $(a_1 b_1) \cdots (a_d b_d)$ forming the path from an element $h \in \mathrm{cyc}(\nu)$ to $(a_1 b_1) \cdots (a_d b_d) h \in \mathrm{cyc}(\mu)$.

We recall the following Lemma from [11, Lemma 2.3]

Lemma 2.1. *Multiplication by $m_\lambda(\mathcal{J})$ defines an endomorphism of $\mathbf{Z}(\mathbf{C}[S_n])$ which, expressed in the $\{C_\mu\}$ basis, is given by*

$$m_\lambda(\mathcal{J}) C_\mu = \sum_{\nu,\, |\nu|=|\mu|} m^\lambda_{\mu\nu} \frac{z_\nu}{|\nu|!} C_\nu, \tag{2.42}$$

where

$$\tilde{m}^\lambda_{\mu\nu} := \frac{|\lambda|!}{\prod_{i=1}^{\ell(|\lambda|} \lambda_i!} m^\lambda_{\mu\nu} \tag{2.43}$$

is the total number of $|\lambda|$*-step paths in the Cayley graph of* S_n *from* $\mathrm{cyc}(\nu)$ *to* $\mathrm{cyc}(\mu)$ *with signature* λ*.*

Combining this with (2.20) gives

$$M_n(q,t,\mathbf{c},z\mathcal{J})\, C_\mu = \sum_{d=0}^{\infty} z^d \sum_{\nu, |\nu|=|\mu|=n} F^d_{M(q,t,\mathbf{c})}(\mu,\nu) z_\nu C_\nu, \tag{2.44}$$

where

$$F^d_{M(q,t,\mathbf{c})}(\mu,\nu) := \frac{1}{|n|!} \sum_{\lambda,\ |\lambda|=d} g_\lambda(\mathbf{c},q,t) m^\lambda_{\mu\nu} \tag{2.45}$$

is the quantum weighted combinatorial Hurwitz number for such paths, and

$$M(q,t,\mathbf{c}) := M(q,t,\mathbf{c},z)|_{z=1}. \tag{2.46}$$

(Note that, whereas the infinite product (2.17) defining the generating function $M(q,t,\mathbf{c},z)$ need not necessarily represent a convergent power series in z, if $|q|<1$ and $|c_i|<1$ for all i, these are i, in fact, convergent for all values of z.)

Combining this with the results of the previous section leads to our first main result: the 2D Toda τ-function defined in (2.36) for $N=0$ is the generating function for the quantum weighted combinatorial Hurwitz number (2.45).

Theorem 2.2. *Expanding* $\tau^{M(q,t,\mathbf{c},z)}(\mathbf{t},\mathbf{s}) := \tau^{M(q,t,\mathbf{c})}(0,\mathbf{t},\mathbf{s})$ *in the basis consisting of products of power sum symmetric functions, the coefficients are the combinatorial quantum Hurwitz numbers (2.45).*

$$\tau^{M(q,t,\mathbf{c},z)}(\mathbf{t},\mathbf{s}) = \sum_{d=0}^{\infty} \sum_{\substack{\mu,\nu \\ |\mu|=|\nu|}} z^d F^d_{M(q,t,\mathbf{c})}(\mu,\nu) p_\mu(\mathbf{t}) p_\nu(\mathbf{s}) \tag{2.47}$$

Proof. Combining (2.44) with (2.23), and using eq. (2.31), gives

$$\sum_{d=0}^{\infty} z^d \sum_{\nu, |\nu|=|\mu|} F^d_{M(q,t,\mathbf{c})}(\mu,\nu)\chi_\lambda(\nu) = \frac{\chi_\lambda(\mu)}{z_\mu} r_\lambda^{M(q,t,\mathbf{c},z)}(N) \tag{2.48}$$

Substituting the Frobenius character formula

$$s_\lambda(\mathbf{t}) = \sum_{\mu,\, |\mu|=|\lambda|} z_\mu^{-1} \chi_\lambda(\mu) p_\mu(\mathbf{t}), \quad s_\lambda(\mathbf{s}) = \sum_{\nu,\, |\mu|=|\lambda|} z_\nu^{-1} \chi_\lambda(\nu) p_\nu(\mathbf{s}), \tag{2.49}$$

and (2.48) into (2.36), for $N=0$, and using the orthogonally of characters, we obtain (2.47). ∎

2.4.2 Enumerative geometrical quantum weighted Hurwitz numbers

We recall the two types of weighting factors appearing in the definition of quantum Hurwitz numbers in ref. [11].

$$\begin{aligned}W_{E(q)}(\mu^{(1)},\dots,\mu^{(k)}) &:= \frac{1}{|\mathrm{aut}(\lambda)|}\sum_{\sigma\in S_k}\sum_{0\le i_1<\cdots<i_k}^{\infty} q^{i_1\ell^*(\mu^{(\sigma(1))})}\cdots q^{i_k\ell^*(\mu^{(\sigma(k))})}\\ &= \frac{1}{|\mathrm{aut}(\lambda)|}\sum_{\sigma\in S_k}\frac{q^{(k-1)\ell^*(\mu^{(\sigma(1))})}\cdots q^{\ell^*(\mu^{(\sigma(k-1))})}}{(1-q^{\ell^*(\mu^{(\sigma(1))})})\cdots(1-q^{\ell^*(\mu^{(\sigma(1))})}\cdots q^{\ell^*(\mu^{(\sigma(k))})})},\end{aligned} \tag{2.50}$$

$$\begin{aligned}W_{H(q)}(\nu^{(1)},\dots,\nu^{(\tilde k)}) &:= \frac{(-1)^{\ell^*(\tilde\lambda)}}{|\mathrm{aut}(\tilde\lambda)|}\sum_{\sigma\in S_{\tilde k}}\sum_{0\le i_1\le\cdots\le i_{\tilde k}}^{\infty} q^{i_1\ell^*(\nu^{(\sigma(1))})}\cdots q^{i_k\ell^*(\nu^{(\sigma(\tilde k))})}\\ &= \frac{(-1)^{\ell^*(\tilde\lambda)}}{|\mathrm{aut}(\tilde\lambda)|}\sum_{\sigma\in S_{\tilde k}}\frac{1}{(1-q^{\ell^*(\nu^{(\sigma(1))})})\cdots(1-q^{\ell^*(\nu^{(\sigma(1))})}\cdots q^{\ell^*(\nu^{(\sigma(\tilde k))})})},\end{aligned} \tag{2.51}$$

where λ is the partition with parts $(\ell^*(\mu^{(1)}),\dots,\ell^*(\mu^{(k)}))$, $\tilde\lambda$ the one with parts $(\ell^*(\nu^{(1)}),\dots,\ell^*(\nu^{(\tilde k)}))$, and $|\mathrm{aut}(\lambda)|$ is the order of the automorphism group of λ :

$$|\mathrm{aut}(\lambda)| = \prod_{i=1}^{\ell(\lambda)} m_i(\lambda)!. \tag{2.52}$$

Denote the product of these

$$W_q(\mu^{(1)},\dots,\mu^{(k)};\nu^{(1)},\dots,\nu^{(\tilde k)}) := W_{E(q)}(\mu^{(1)},\dots,\mu^{(k)})W_{H(q)}(\nu^{(1)},\dots,\nu^{(\tilde k)}). \tag{2.53}$$

Recall also the definition of the Pochhammer symbol $(u)_\lambda$ associated with a partition λ

$$(u)_\lambda := \prod_{i=1}^{\ell(\mu)}\prod_{j=1}^{\lambda_i}(u+j-i) \tag{2.54}$$

and the following Lemma (cf. [26]), which follows from the Frobenius character formula.

Lemma 2.3. *The Pochhammer symbol may be expressed as*

$$(u)_\lambda = s_\lambda(\mathbf{t}(u))h_\lambda = \left(1 + h_\lambda \sum_{\mu,\,|\mu|=|\lambda|}{}' \frac{\chi_\lambda(\mu)}{z_\mu}u^{-\ell^*(\mu)}\right) \tag{2.55}$$

where

$$\mathbf{t}(u) := (u, \frac{u}{2}, \frac{u}{3}, \dots), \tag{2.56}$$

and $\sum'_{\mu,|\mu|=|\lambda|)}$ *denotes the sum over all partitions other than the cycle type of the identity element* $(1)^{|\lambda|}$.

It is useful to know how any given symmetric combination of the Jucys-Murphy elements may be represented in the basis of cycle-type sums (see e.g. [19]). The following result shows how to do this for the symmetric functions $g_j(\mathcal{J},q,t)$.

Theorem 2.4.

$$g_j(\mathcal{J},q,t)=\sum_{e=0}^{j}t^e\sum_{k,\tilde{k}=0}^{e,j-e}\sideset{}{'}\sum_{\substack{\{\mu^{(u)},\nu^{(v)}\}\\ 1\le u\le k,1\le v\le\tilde{k}\\ |\mu^{(u)}|=|\nu^{(v)}|=n\\ \sum_{u=1}^{k}\ell^*(\mu^{(u)})=e\\ \sum_{u=1}^{k}\ell^*(\mu^{(u)})+\sum_{v=1}^{\tilde{k}}\ell^*(\nu^{(v)})=j}}W_q(\mu^{(1)},\dots,\mu^{(k)};\nu^{(1)},\dots,\nu^{(\tilde{k})})\prod_{u=1}^{k}C_{\mu^{(u)}}\prod_{v=1}^{\tilde{k}}C_{\nu^{(v)}}\quad(2.57)$$

where, by (2.40),

$$\prod_{u=1}^{k}C_{\mu^{(u)}}\prod_{v=1}^{\tilde{k}}C_{\nu^{(v)}}=\sum_{\nu,\,|\nu|=n}H(\mu^{(1)},\dots,\mu^{(k)},\nu^{(1)},\dots,\nu^{(\tilde{k})},\nu)\,z_\nu C_\nu\tag{2.58}$$

Remark 2.1. Note that the sums appearing in (2.57) are all finite because of the fact that the partitions corresponding to the identity class of S_n are excluded and the constraints

$$\sum_{u=1}^{k}\ell^*(\mu^{(u)})=e,\quad\sum_{u=1}^{k}\ell^*(\mu^{(u)})+\sum_{v=1}^{\tilde{k}}\ell^*(\nu^{(v)})=j\tag{2.59}$$

imply the number of partitions $k+\tilde{k}$ is finite.

Proof. We start with the expansion

$$\prod_{a=1}^{n}\prod_{k=0}^{\infty}\frac{1-tz\mathcal{J}_aq^k}{1-z\mathcal{J}_aq^k}=\sum_{j=0}^{\infty}g_j(\mathcal{J},q,t)z^j\tag{2.60}$$

Applying the LHS of (2.60) to F_λ and using (2.30) gives

$$\prod_{a=1}^{n}\prod_{k=0}^{\infty}\frac{1-tz\mathcal{J}_aq^k}{1-z\mathcal{J}_aq^k}F_\lambda=\prod_{(i,j)\in\lambda}\prod_{k=0}^{\infty}\frac{1-tz(j-i)q^k}{1-z(j-i)q^k}F_\lambda=\prod_{k=0}^{\infty}\frac{(-\frac{1}{tzq^k})_\lambda}{(-\frac{1}{zq^k})_\lambda}F_\lambda\tag{2.61}$$

$$=\prod_{k=0}^{\infty}\frac{1+h_\lambda\sideset{}{'}\sum_{\mu,\,|\mu|=|\lambda|}\frac{\chi_\lambda(\mu)}{z_\mu}(-tzq^k)^{\ell^*(\mu)}}{1+h_\lambda\sideset{}{'}\sum_{\nu,\,|\nu|=|\lambda|}\frac{\chi_\lambda(\nu)}{z_\nu}(-zq^k)^{\ell^*(\nu)}}F_\lambda,\tag{2.62}$$

where Lemma 2.3 has been used in both the numerator and denominator of (2.62). From the relation (2.26) and the fact that $\{F_\lambda\}$ is a basis for the center $\mathbf{Z}(\mathbf{C}[S_n])$, eq. (2.62), together with (2.60), is equivalent to the identity

$$\sum_{j=0}^{\infty}g_j(\mathcal{J},q,t)z^j=\prod_{k=0}^{\infty}\frac{1+\sum C_\mu(-tzq^k)^{\ell^*(\mu)}}{1+\sum C_\nu(-zq^k)^{\ell^*(\nu)}}.\tag{2.63}$$

Expanding (2.63) as a power series in z and t, and summing the resulting geometric series expansions in q, as detailed in [11], to obtain (2.50) and (2.51) gives the result (2.57). $\square$

Now let $\{\{\mu^{i,u_i}\}_{u_i=1,\dots,k_i}, \{\nu^{i,v_i}\}_{v_i=1,\dots,\tilde{k}_i}, \mu, \nu\}_{i=1,\dots,l}$ denote the branching profiles of an n-sheeted covering of the Riemann sphere with two specified branch points of ramification profile types (μ,ν), at $(0,\infty)$, and the rest divided into two classes I and II, denoted $\{\mu^{(i,u_i)}\}_{u_i=1,\dots,k_i}$ and $\{\nu^{(i,v_i)}\}_{v_i=1,\dots,\tilde{k}_i}$, respectively. These are further subdivided into l species, or "colours", labelled by $i=1,\dots l$, the elements within each colour group distinguished by the labels $(u_i=1,\dots,k_i)$ and $(v_i=1,\dots,\tilde{k}_i)$. To such a grouping, we assign a partition λ of length

$$\ell(\lambda) = l \tag{2.64}$$

and weight

$$d := |\lambda| = \sum_{i=1}^{l}\left(\sum_{u_i=1}^{k_i}\ell^*(\mu^{(i,u_i)}) + \sum_{v_i=1}^{\tilde{k}_i}\ell^*(\nu^{(i,v_i)})\right) = \sum_{i=1}^{l} d_i, \tag{2.65}$$

whose parts $(\lambda_1 \geq \dots \geq \lambda_l > 0)$ are equal the total colengths

$$d_i := \sum_{u_i=1}^{k_i}\ell^*(\mu^{(i,u_i)}) + \sum_{v_i=1}^{\tilde{k}_i}\ell^*(\nu^{(i,v_i)}), \quad i = 1.\dots,l \tag{2.66}$$

in weakly decreasing order. By the Riemann-Hurwitz formula, the genus g of the covering curve is given by

$$2-2g = \ell(\mu)+\ell(\nu) - d. \tag{2.67}$$

We now assign a weight $W_q(\{\mu^{(i,u_i)},\nu^{(i,v_i)}\},\mathbf{c})$ to each such covering, consisting of the product of all the weights $W_{E(q)}(\{\mu^{(i,u_i)}\}_{u_i=1,\dots,k_i})$, $W_{H(q)}(\{\nu^{(i,v_i)}\}_{v_i=1,\dots,\tilde{k}_i})$ for the subsets of different colour and class and the weight $m_\lambda(\mathbf{c})$ given by the monomial symmetric functions evaluated at the parameters $\mathbf{c}$

$$W_q(\{\mu^{(i,u_i)},\nu^{(i,v_i)}\},\mathbf{c}) := W_q(\{\mu^{(i,u_i)},\nu^{(i,v_i)}\})m_\lambda(\mathbf{c}) \tag{2.68}$$

$$W_q(\{\mu^{(i,u_i)},\nu^{(i,v_i)}\}) := \prod_{i=1}^{l} W_{E(q)}(\{\mu^{(i,u_i)}\}_{u_i=1,\dots,k_i}) W_{H(q)}(\{\nu^{(i,v_i)}\}_{i=1,\dots,\tilde{k}_i}) \tag{2.69}$$

Using these weights, for every pair (d,e) of non-negative integers and (μ,ν) of partitions of n, we define the geometrical quantum weighted Hurwitz numbers $H^{(d,e)}_{(\mathbf{c},q)}(\mu,\nu)$ as the sum

$$H^{(d,e)}_{(\mathbf{c},q)}(\mu,\nu) := z_\nu \sum_{l=0}^{d} \sideset{}{'}\sum_{\substack{\{\mu^{(i,u_i)},\nu^{(i,v_i)}\},\ k_i\geq 1,\ \tilde{k}_i\geq 1 \\ \sum_{i=1}^{l}\sum_{u_i=1}^{k_i}\ell^*(\mu^{(i,u_i)})=e, \\ \sum_{i=1}^{l}\left(\sum_{u_i=1}^{k_i}\ell^*(\mu^{(i,u_i)})+\sum_{v_i=1}^{\tilde{k}_i}\ell^*(\nu^{(i,v_i)})\right)=d}} W_q(\{\mu^{(i,u_i)},\nu^{(i,v_i)}\},\mathbf{c})\, H(\{\mu^{(i,u_i)}\}_{\substack{u_i=1,\dots,k_i \\ i=1,\dots,l}}, \{\nu^{(i,v_i)}\}_{\substack{v_i=1,\dots,\tilde{k}_i \\ i=1,\dots,l}}, \mu,\nu). \tag{2.70}$$

Theorem 2.5. *The combinatorial Hurwitz numbers $F^d_{M(q,t,\mathbf{c})}(\mu,\nu)$ are polynomials in t of degree d whose coefficients are equal to the geometrical quantum weighted Hurwitz numbers $H^{(d,e)}_{(\mathbf{c},q)}(\mu,\nu)$*

$$F^d_{M(q,t,\mathbf{c})}(\mu,\nu)=\sum_{e=0}^{d} H^{(d,e)}_{(\mathbf{c},q)}(\mu,\nu)t^e. \tag{2.71}$$

Hence $\tau^{M(q,t,\mathbf{c},z)}(\mathbf{t},\mathbf{s})$, when expanded in the basis of products of power sum symmetric functions and power series in z and t is the generating function for the $H^{(d,e)}_{(\mathbf{c},q)}(\mu,\nu)$'s:

$$\tau^{M(q,t,\mathbf{c},z)}(\mathbf{t},\mathbf{s})=\sum_{d=0}^{\infty}\sum_{e=0}^{d} z^d t^e H^{(d,e)}_{(\mathbf{c},q)}(\mu,\nu)p_\mu(\mathbf{t})p_\nu(\mathbf{s}). \tag{2.72}$$

Proof. Substitution of (2.57) into

$$g_\lambda(\mathcal{J},q,t)=\prod_{i=1}^{\ell(\lambda)} g_{\lambda_i}(\mathcal{J},q,t) \tag{2.73}$$

gives

$$g_\lambda(\mathcal{J},q,t)=\sum_{e=0}^{|\lambda|} t^e \sideset{}{'}\sum_{\substack{\{\mu^{(i,u_i)},\nu^{(i,v_i)}\} \\ \sum_{i=1}^{\ell(\lambda)}\sum_{u_i=1}^{k_i}\ell^*(\mu^{(i,u_i)})=e, \\ \sum_{u_i=1}^{k_i}\ell^*(\mu^{(i,u_i)})+\sum_{v_i=1}^{\tilde{k}_i}\ell^*(\nu^{(i,v_i)})=\lambda_i}} W_q(\{\mu^{(i,u_i)},\nu^{(i,v_i)}\})\prod_{i=1}^{\ell(\lambda)}\left(\prod_{u_i=1}^{k_i} C_{\mu^{(i,u_i)}}\prod_{v_i=1}^{\tilde{k}_i} C_{\nu^{(i,v_i)}}\right) \tag{2.74}$$

Combining this with (2.21) gives

$$M_n(q,t,\mathbf{c},z\mathcal{J})C_\mu=\sum_{d=0}^{\infty}\sum_{e=0}^{d} z^d t^e \sum_{\substack{\nu \\ |\nu|=|\mu|=n}} H^{(d,e)}_{(\mathbf{c},q)}(\mu,\nu)C_\nu, \tag{2.75}$$

where $H^{(d,e)}_{(\mathbf{c},q)}(\mu,\nu)$ is defined by (2.70). Comparing with (2.44) gives the result (2.71), and hence (2.72). □

3 Specializations, limits and examples

By making specific choices for the parameters $\{(c_1,c_2,\dots),q,t\}$ defining the weight generating function $M(q,t,\mathbf{c},z)$, specialized versions of the above quantum weighted Hurwitz numbers result. Taking the limits $(z,t)\to(0,\infty)$, with tz fixed gives the quantum deformation of the path weighting by elementary symmetric functions considered in [11]. The limit $t\to 0$ gives the dual case, weighted by the quantum deformation of the path weighting

by complete symmetric functions. Other specializations involving only particular values for the pair (q,t) or their limits reduce the Macdonald polynomials either to Schur polynomials $(q=t)$, or Hall-Littlewood polynomials $(q=0)$ or Jack polynomials $(q=t^\alpha,\ t\to 1)$. In this way we can recover all previously studied versions of weighted Hurwitz numbers, as well as several new examples of interest.

3.1 Classically weighted Hurwitz numbers $(q=t)$

By setting $t=q$ in (2.19) we recover the case of Schur functions and the general classically weighted families of Hurwitz numbers studied in [11].

3.2 The case $c_i=-\delta_{i,1}$ (quantum monotonic paths)

This gives the quantum deformation of the classical case (corresponding to $q=0$) when the weight generating function is $\frac{1+w}{1-z}$, with $w=-tz$. If $w=0$, the latter becomes the signed counting problem for branched covers with fixed genus or, equivalently, weakly monotonic paths in the Cayley graph generated by transpositions [10, 11]. When $z=0$ it gives the Hurwitz numbers for Belyi curves (having three branch points, with two of them fixed) of fixed genus or, equivalently, strongly monotonic paths in the Cayley graph generated by transpositions [10, 11]. When $q\neq 0$, $t=1$, this is the particular case of the multispecies quantum Hurwitz numbers $F^d_{Q(q,q)}(\mu,\nu)=H^d_{Q(q,q)}(\mu,\nu)$ developed in detail in [11], when there are only two species involved, one of the first class, the other, of second.

3.3 Elementary quantum weighting ($(z,t)\to(0,\infty)$, $-tz$ fixed ($\to z$)

For this case, the weight generating function is

$$E(q,\mathbf{c},z) := \prod_{k=0}^{\infty}\prod_{i=1}^{\infty}(1+zq^k c_i) = \prod_{i=1}^{\infty}(-zc_i;q)_\infty =: \sum_{j=0}^{\infty} e_j(q,\mathbf{c})z^j, \tag{3.1}$$

where $e_j(q,\mathbf{c})$ is the quantum deformation of the elementary symmetric function $e_j(\mathbf{c})$ (the classical limit being $q\to 0$). Setting $c_i=\delta_{i1}$ reproduces the generating function functions for the special quantum weighted Hurwitz numbers denoted $H^d_{E(q)}(\mu,\nu)=F^d_{E(q)}(\mu,\nu)$ that were studied in [11].

In the general case, the corresponding element of the center of the group algebra is:

$$E_n(q,\mathbf{c},z\mathcal{J}) := \prod_{a=1}^{n} E(q,\mathbf{c},z\mathcal{J}_a) = \sum_{\lambda} z^{|\lambda|}e_\lambda(q,\mathbf{c})m_\lambda(\mathcal{J}) = \sum_{\lambda} z^{|\lambda|}m_\lambda(\mathcal{J})e_\lambda(q,\mathbf{c}) \tag{3.2}$$

where

$$e_\lambda(q,\mathbf{c}) := \prod_{i=1}^{\ell(\lambda)} e_{\lambda_i}(q,\mathbf{c}). \tag{3.3}$$

Applying $E_n(q, \mathbf{c}, z\mathcal{J}) \in \mathbf{Z}(\mathbf{C}[S_n])$ to the orthogonal idempotents $\{F_\lambda\}$ and the cycle-type sums $\{C_\mu\}$, it follows that the corresponding hypergeometric $2D$ Toda τ-function is

$$\tau^{E(q,\mathbf{c},z)}(\mathbf{t}, \mathbf{s}) = \sum_\lambda r_\lambda^{E(q,\mathbf{c},z)} s_\lambda(\mathbf{t}) s_\lambda(\mathbf{s}) \tag{3.4}$$

$$= \sum_{d=0}^\infty z^d \sum_\lambda F^d_{E(q,\mathbf{c})}(\mu, \nu) p_\mu(\mathbf{t}) p_\nu(\mathbf{s}), \tag{3.5}$$

where the content product coefficient $r_\lambda^{E(q,\mathbf{c},z)}$ is

$$r_\lambda^{E(q,\mathbf{c},z)} := \prod_{(ij)\in\lambda} \prod_{k=0}^\infty (-z(j-i)c_k; q)_\infty \tag{3.6}$$

and

$$F^d_{E(q,\mathbf{c})}(\mu, \nu) := \sum_{|\lambda|=d} e_\lambda(q, \mathbf{c}) m^\lambda_{\mu\nu} \tag{3.7}$$

is the weighted number of paths in the Cayley graph of S_n generated by transpositions, starting at the conjugacy class $\mathrm{cyc}(\mu)$ and ending at $\mathrm{cyc}(\nu)$, with the weight $e_\lambda(q, \mathbf{c})$ for a path of signature λ.

Now consider n-fold branched coverings of $\mathbf{CP}^1$ with a fixed pair of branch points at $(0, \infty)$ with ramification profiles (μ, ν) and a further $\sum_{i=1}^l k_i$ branch points $\{\mu^{(i,u_i)}\}_{u_i=1,\dots,k_i}$ of l different species (or "colours"), labelled by $i = 1, \dots, l$, with non trivial ramification profiles. The weight $W_{E^l(q)}(\{\mu^{(i,u_i)}\}_{\substack{u_i=1,\dots,k_i \\ i=1,\dots,l}}, \mathbf{c})$ for such a covering consists of the product of all the weights $W_{E(q)}(\{\mu^{(i,u_i)}\}_{u_i=1,\dots,k_i})$, for the subsets of different colour with the weight $m_\lambda(\mathbf{c})$ given by the monomial symmetric functions evaluated at the parameters $\mathbf{c}$

$$W_{E^l(q)}(\{\mu^{(i,u_i)}\}_{\substack{u_i=1,\dots,k_i \\ i=1,\dots,l}}, \mathbf{c}) := W_{E^l(q)}(\{\mu^{(i,u_i)}\}_{\substack{u_i=1,\dots,k_i \\ i=1,\dots,l}})\, m_\lambda(\mathbf{c}) \tag{3.8}$$

$$W_{E^l(q)}(\{\mu^{(i,u_i)}\}_{\substack{u_i=1,\dots,k_i \\ i=1,\dots,l}}) := \prod_{i=1}^l W_{E(q)}(\{\mu^{(i,u_i)}\}_{u_i=1,\dots,k_i}). \tag{3.9}$$

We then have

$$F^d_{E(q,\mathbf{c})}(\mu, \nu) = H^d_{E(q,\mathbf{c})}(\mu, \nu), \tag{3.10}$$

where

$$H^d_{E(\mathbf{c},q)}(\mu, \nu) := z_\nu \sum_{l=0}^d \sideset{}{'}\sum_{\substack{\{\mu^{(i,u_i)}\},\ k_i\geq 1, \\ \sum_{i=1}^l \sum_{u_i=1}^{k_i} \ell^*(\mu^{(i,u_i)})=d}} W_{E^l(q)}(\{\mu^{(i,u_i)}\}_{\substack{u_i=1,\dots,k_i \\ i=1,\dots,l}}, \mathbf{c})\ H(\{\mu^{(i,u_i)}\}_{\substack{u_i=1,\dots,k_i \\ i=1,\dots,l}}, \mu, \nu) \tag{3.11}$$

is the geometrical elementary quantum weighted Hurwitz number.

3.4 Complete quantum weighting ($t = 0$)

This is the dual of the preceding case, with weight generating function

$$H(q, \mathbf{c}, z) := \prod_{k=0}^{\infty} \prod_{i=1}^{\infty} (1 - zq^k c_i)^{-1} = \prod_{i=1}^{\infty} (zc_i; q)_{\infty}^{-1} =: \sum_{j=0}^{\infty} h_j(q, \mathbf{c}) z^j, \tag{3.12}$$

where $h_j(q, \mathbf{c})$ is the quantum deformation of the complete symmetric function $h_j(\mathbf{c})$. Setting $c_i = \delta_{i1}$ reproduces the generating function functions for the quantum weighted Hurwitz numbers $H^d_{H(q)}(\mu, \nu) = F^d_{H(q)}(\mu, \nu)$ studied in [11]. The corresponding element of the center of the group algebra in the general case is:

$$H_n(q, \mathbf{c}, z\mathcal{J}) := \prod_{a=1}^{n} H(q, \mathbf{c}, \mathcal{J}_a) \ = \sum_{\lambda} z^{|\lambda|} h_\lambda(q, \mathbf{c}) m_\lambda(\mathcal{J}) = \sum_{\lambda} z^{|\lambda|} m_\lambda(\mathbf{c}) h_\lambda(q, \mathcal{J}), \tag{3.13}$$

where

$$h_\lambda(q, \mathbf{c}) := \prod_{i=1}^{\ell(\lambda)} h_{\lambda_i}(q, \mathbf{c}). \tag{3.14}$$

The hypergeometric $2D$ Toda τ-function for this case is

$$\tau^{H(q,\mathbf{c},z)}(\mathbf{t}, \mathbf{s}) = \sum_{\lambda} r_\lambda^{H(q,\mathbf{c},z)} s_\lambda(\mathbf{t}) s_\lambda(\mathbf{s}) \tag{3.15}$$

$$= \sum_{d=0}^{\infty} z^d \sum_{\lambda} F^d_{H(q,\mathbf{c})}(\mu, \nu) p_\mu(\mathbf{t}) p_\nu(\mathbf{s}), \tag{3.16}$$

where

$$r_\lambda^{H(q,\mathbf{c},z)} := \prod_{(ij)\in\lambda} \prod_{k=0}^{\infty} (z(j-i)c_k; q)_{\infty}^{-1} \tag{3.17}$$

and

$$F^d_{H(q,\mathbf{c})}(\mu, \nu) := \sum_{|\lambda|=d} h_\lambda(q, \mathbf{c}) m^\lambda_{\mu\nu} \tag{3.18}$$

is the weighted number of paths in the Cayley graph of S_n generated by transpositions, starting at the conjugacy class $\mathrm{cyc}(\mu)$ and ending at $\mathrm{cyc}(\nu)$, with weight $h_\lambda(q, \mathbf{c})$ for a path of signature λ.

Consider again n-fold branched coverings of $\mathbf{CP}^1$, with a fixed pair of branch points at $(0, \infty)$ with ramification profiles (μ, ν) and a further $\sum_{i=1}^{l} \tilde{k}_i$ branch points $\{\nu^{(i,v_i)}\}_{v_i=1,\dots,\tilde{k}_i}$ again, of l different species (or "colours"), labelled by $i = 1, \dots, l$, with nontrivial ramification profiles. Like in the preceding case, the weight $W_{H^l(q)}(\{\nu^{(i,v_i)}\}_{\substack{v_i=1,\dots,\tilde{k}_i \\ i=1,\dots,l}}, \mathbf{c})$ for such a covering consists now of the product of all weights $W_{H(q)}(\{\nu^{(i,v_i)}\}_{v_i=1,\dots,\tilde{k}_i})$, for the subsets of different

colour with the weight $m_\lambda(\mathbf{c})$ again given by the monomial symmetric functions evaluated at the parameters $\mathbf{c}$

$$W_{H^l(q)}(\{\nu^{(i,v_i)}\}_{\substack{v_i=1,\dots,\tilde{k}_i\\ i=1,\dots,l}}, \mathbf{c}) := W_{H^l(q)}(\{\nu^{(i,v_i)}\}_{\substack{v_i=1,\dots,\tilde{k}_i\\ i=1,\dots,l}})\, m_\lambda(\mathbf{c}) \tag{3.19}$$

$$W_{H^l(q)}(\{\nu^{(i,v_i)}\}_{\substack{v_i=1,\dots,\tilde{k}_i\\ i=1,\dots,l}}) := \prod_{i=1}^{l} W_{H(q)}(\{\nu^{(i,v_i)}\}_{v_i=1,\dots,\,\tilde{k}_i}). \tag{3.20}$$

We again have the equality

$$F^d_{H(q,\mathbf{c})}(\mu,\nu) = H^d_{H(q,\mathbf{c})}(\mu,\nu), \tag{3.21}$$

where

$$H^d_{H(\mathbf{c},q)}(\mu,\nu) := z_\nu \sum_{l=0}^{d} \sideset{}{'}\sum_{\substack{\{\nu^{(i,v_i)}\},\ \tilde{k}_i\geq 1,\\ \sum_{i=1}^{l}\sum_{v_i=1}^{\tilde{k}_i} \ell^*(\nu^{(i,v_i)})=d}} W_{H^l(q)}(\{\nu^{(i,v_i)}\}_{\substack{v_i=1,\dots,\tilde{k}_i\\ i=1,\dots,l}}, \mathbf{c})\ H(\{\nu^{(i,v_i)}\}_{\substack{v_i=1,\dots,\tilde{k}_i\\ i=1,\dots,l}}, \mu,\nu) \tag{3.22}$$

is the corresponding geometrically defined complete quantum weighted Hurwitz number.

3.5 Hall-Littlewood polynomials ($q = 0$)

Setting $q = 0$ in eq. (2.8), the generating function reduces to the one for Hall-Littlewood polynomials [21, Sec. III.2] $P_\lambda(\mathbf{x}, t)$, which satisfy the orthogonality relations

$$(P_\lambda, P_\mu)_t = \delta_{\lambda\mu}(b_\lambda(t))^{-1}, \quad b_\lambda(t) := \prod_{i\geq 1}\prod_{k=1}^{m_i(\lambda)}(1-t^k) \tag{3.23}$$

with respect to the scalar product $(\ ,\ )_t$ defined by

$$(p_\lambda, p_\mu)_t = \delta_{\lambda\mu} z_\lambda n_\lambda(t), \quad n_\lambda := \prod_{i=1}^{\ell(\lambda)} \frac{1}{1-t^{\lambda_i}}. \tag{3.24}$$

Following [21], we define

$$q_\lambda(\mathbf{x}, t) := b_\lambda(t) \prod_{i=1}^{\ell(\lambda)} P_j(\mathbf{x}, t) \tag{3.25}$$

and obtain the following expansion

$$L(t, \mathbf{x}, \mathbf{y}) := \prod_{i,j}^{\infty} \frac{1 - t x_i y_j}{1 - x_i y_j} = \sum_{\lambda}^{\infty} q_\lambda(\mathbf{x}, t) m_\lambda(\mathbf{y}) = \sum_{\lambda}^{\infty} q_\lambda(\mathbf{y}, t) m_\lambda(\mathbf{x}). \tag{3.26}$$

Substituting $\mathbf{c} = (c_1, c_2, \dots)$ for $\mathbf{x}$, and $(\mathcal{J}_1, \dots, \mathcal{J}_n)$ for $\mathbf{y}$, we have

$$L(t, \mathbf{c}, z\mathcal{J}) := \prod_{i=1}^{\infty}\prod_{a=1}^{n} \frac{1 - tc_i z\mathcal{J}_a}{1 - c_i z\mathcal{J}_a} = \sum_{\lambda}^{\infty} z^{|\lambda|} q_\lambda(\mathbf{c}, t) m_\lambda(\mathcal{J}) = \sum_{\lambda}^{\infty} z^{|\lambda|} q_\lambda(\mathcal{J}, t) m_\lambda(\mathbf{c}). \tag{3.27}$$

Applying $L(t, \mathbf{c}, z\mathcal{J}), \in \mathbf{Z}(\mathbf{C}[S_n])$ to the orthogonal idempotents $\{F_\lambda\}$ and the cycle-type sums $\{C_\mu\}$ as above, the corresponding hypergeometric $2D$ Toda τ-function becomes

$$\tau^{L(t,\mathbf{c},z)}(\mathbf{t}, \mathbf{s}) = \sum_{\lambda} r_\lambda^{L(t,\mathbf{c},z)} s_\lambda(\mathbf{t}) s_\lambda(\mathbf{s}) \tag{3.28}$$

$$= \sum_{d=0}^{\infty} z^d \sum_{\lambda} F^d_{L(t,\mathbf{c})}(\mu, \nu) p_\mu(\mathbf{t}) p_\nu(\mathbf{s}) \tag{3.29}$$

$$\tag{3.30}$$

where

$$r_\lambda^{L(t,\mathbf{c},z)} := \prod_{(ij)\in\lambda} \prod_{k=1}^{\infty} \frac{1 - tz(j-i)c_k}{1 - z(j-i)c_k} = \prod_{k=1}^{\infty} (-t)^{|\lambda|} \frac{(-1/(tzc_k))_\lambda}{(-1/(zc_k))_\lambda} \tag{3.31}$$

and

$$F^d_{L(t,\mathbf{c})}(\mu, \nu) := \sum_{|\lambda|=d} q_\lambda(\mathbf{c}, t) m^\lambda_{\mu\nu} \tag{3.32}$$

is again the weighted number of paths in the Cayley graph of S_n generated by transpositions, with weight $q_\lambda(\mathbf{c}, t)$ for a path of signature λ.

We also have

$$F^d_{L(t,\mathbf{c})}(\mu, \nu) = \sum_{e=0}^{d} H^{(d,e)}_{L(\mathbf{c})}(\mu, \nu) t^e \tag{3.33}$$

where

$$H^{(d,e)}_{L(\mathbf{c})}(\mu, \nu) := z_\nu \sum_{l=0}^{d} \sideset{}{'}\sum_{\substack{\{\mu^{(i,u_i)}, \nu^{(i,v_i)}\},\ k_i \ge 1,\ \tilde{k}_i \ge 1 \\ \sum_{i=1}^{l} \sum_{u_i=1}^{k_i} \ell^*(\mu^{(i,u_i)}) = e, \\ \sum_{i=1}^{l} \left(\sum_{u_i=1}^{k_i} \ell^*(\mu^{(i,u_i)}) + \sum_{v_i=1}^{\tilde{k}_i} \ell^*(\nu^{(i,v_i)}) \right) = d}} (-1)^{K+d-e} H(\{\mu^{(i,u_i)}\}_{\substack{u_i=1,\dots,k_i \\ i=1,\dots,l}}, \{\nu^{(i,v_i)}\}_{\substack{v_i=1,\dots,\tilde{k}_i \\ i=1,\dots,l}}, \mu, \nu), \tag{3.34}$$

with

$$K := \sum_{i=1}^{l} (k_i + \tilde{k}_i) \tag{3.35}$$

the total number of branch points. $H^{(d,e)}_{\mathbf{c}}(\mu, \nu)$ is the weighted generalization of the multi-species hybrid signed Hurwitz numbers studied in [15]. As in the general Macdonald case, $H^{(d,e)}_{L(\mathbf{c})}(\mu, \nu)$ is the weighted number of n-fold branched coverings of $\mathbf{CP}^1$ with a fixed pair of

branch points with ramification profiles (μ, ν), and K additional branch points divided into two classes I and II, denoted $\{\mu^{(i,u_i)}\}_{u_i=1,\dots,k_i}$ and $\{\nu^{(i,v_i)}\}_{v_i=1,\dots,\tilde{k}_i}$, respectively, which are further subdivided into l species, or "colours", labelled by $i = 1, \dots l$, the elements within each colour group distinguished by the labels $(u_i = 1, \dots, k_i)$ and $(v_i = 1, \dots, \tilde{k}_i)$. To such a grouping, we again assign a partition λ of length

$$\ell(\lambda) = l \tag{3.36}$$

and weight

$$d := |\lambda| = \sum_{i=1}^{l} \left(\sum_{u_i=1}^{k_i} \ell^*(\mu^{(i,u_i)}) + \sum_{v_i=1}^{\tilde{k}_i} \ell^*(\nu^{(i,v_i)}) \right) = \sum_{i=1}^{l} d_i, \tag{3.37}$$

whose parts $(\lambda_1 \geq \cdots \geq \lambda_l > 0)$ are equal the total colengths

$$d_i := \sum_{u_i=1}^{k_i} \ell^*(\mu^{(i,u_i)}) + \sum_{v_i=1}^{\tilde{k}_i} \ell^*(\nu^{(i,v_i)}), \quad i = 1 \dots, l \tag{3.38}$$

in weakly decreasing order.

3.6 Jack polynomials ($q = t^\alpha$, $t \to 1$)

Setting $q = t^\alpha$ and taking the limit $q \to 1$, we obtain the Jack polynomials [21, Sec. VI.10] $P_\lambda^{(\alpha)}$ as the limiting case of the MacDonald polynomials. These satisfy the orthogonality relations

$$\langle P_\lambda^\alpha, P_\mu^\alpha \rangle_\alpha = \delta_{\lambda\mu} z_\lambda (b_\lambda^{(\alpha)})^{-1}, \quad b_\lambda^{(\alpha)} := \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} \frac{\alpha(\lambda_i - j) + \lambda'_j - i + 1}{\alpha(\lambda_i - j) + \lambda_j - i + \alpha} \tag{3.39}$$

with respect to the scalar product $\langle \ , \ \rangle_\alpha$ defined by

$$\langle p_\lambda, p_\mu \rangle_\alpha = \delta_{\lambda\mu} z_\lambda \alpha^{\ell(\lambda)}. \tag{3.40}$$

This corresponds to the family of weight generating functions

$$J(\alpha, \mathbf{c}, z) := \prod_{k=1}^{\infty} (1 - z c_i)^{-1/\alpha} \tag{3.41}$$

and the corresponding family of central elements

$$J(\alpha, \mathbf{c}, z\mathcal{J}) := \prod_{i=1}^{\infty} \prod_{a=1}^{n} (1 - z c_i \mathcal{J}_a)^{-1/\alpha} = \sum_\lambda z^{|\lambda|} g_\lambda^\alpha(\mathcal{J}) m_\lambda(\mathbf{c}) = \sum_\lambda z^{|\lambda|} g_\lambda^\alpha(\mathbf{c}) m_\lambda(\mathcal{J}), \tag{3.42}$$

where the symmetric functions $g^{\alpha}_{\lambda}(\mathbf{x})$ are the analogs of the $e_{\lambda}(\mathbf{x})$ or $h_{\lambda}(\mathbf{x})$ bases formed from products of elementary or complete symmetric functions in the case of Schur functions ($\alpha = 1$),

$$g^{\alpha}_{\lambda}(\mathbf{x}) = \alpha^{\ell(\lambda)} \prod_{i=1}^{\ell(\lambda} P^{(\alpha)}_{(\lambda_i)}(\mathbf{x}), \tag{3.43}$$

The content product coefficients entering in the double Schur function expansion of the associated hypergeometric $2D$ Toda τ-functions

$$\tau^{J(\alpha,\mathbf{c},z)}(\mathbf{t},\mathbf{s}) = \sum_{\lambda} r_{\lambda}^{J(\alpha,\mathbf{c},z)} s_{\lambda}(\mathbf{t}) s_{\lambda}(\mathbf{s}) \tag{3.44}$$

in this case are

$$r_{\lambda}^{J(\alpha,\mathbf{c},z)} = \prod_{(ij)\in\lambda} \prod_{k=0}^{\infty} (1 - z(j-i)c_k)^{-1/\alpha} = \prod_{k=1}^{\infty} (1 - zc_k)^{\frac{|\lambda|}{\alpha}} (-1/zc_k)_{\lambda}^{-1/\alpha}. \tag{3.45}$$

The expansion in the basis of products of power sum symmetric functions is therefore

$$\tau^{J(\alpha,\mathbf{c},z)}(\mathbf{t},\mathbf{s}) = \sum_{d=0}^{\infty} \sum_{\substack{\mu,\nu \\ |\mu|=|\nu|=n}} z^d F^d_{J(\alpha,\mathbf{c})}(\mu,\nu) p_{\mu}(\mathbf{t}) p_{\nu}(\mathbf{s}) \tag{3.46}$$

where

$$F^d_{J(\alpha,\mathbf{c})}(\mu,\nu) = \sum_{\lambda} g^{\alpha}_{\lambda}(\mathbf{c}) m^{\lambda}_{\mu\nu} \tag{3.47}$$

is the combinatorial Hurwitz number giving the weighted number of d-step paths of signature λ in the Cayley graph of S_n , starting in the conjugacy class $\mathrm{cyc}(\mu)$ and ending in $\mathrm{cyc}(\nu)$, with weight $g^{\alpha)}_{\lambda}(\mathbf{c})$.

We again have

$$F^d_{J(\alpha,\mathbf{c})}(\mu,\nu) = H^d_{J(\alpha,\mathbf{c})}(\mu,\nu), \tag{3.48}$$

where the weighted geometrical Hurwitz number is

$$H^d_{J(\alpha,\mathbf{c})}(\mu,\nu) := \sum_{k=0}^{\infty} \binom{-\frac{1}{\alpha}}{k} \sum_{\substack{\mu^{(1)},\dots,\mu^{((k)} \\ |\mu^{(i)}|=n \\ \sum_{i=1}^{k} \ell^*(\mu^{(i)})=d}} m_{\lambda}(\mathbf{c}) H(\mu^{(1)},\dots,\mu^{(k)},\mu,\nu), \tag{3.49}$$

with the sum over partitions λ of length k, and weight d whose parts are $\{\ell^*(\mu^{(1)}),\dots,\ell^*(\mu^{(k)})\}$.

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